NOTE ON THE SET OF BRAGG PEAKS WITH HIGH INTENSITY

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ABSTRACT. We consider diffraction of Delone sets in Euclidean space. We show that the set of Bragg peaks with high intensity is always Meyer (if it is relatively dense). We use this to provide a new characterization for Meyer sets in terms of positive and positive definite measures. Our results are based on a careful study of positive definite measures, which may be of interest in its own right.

INTRODUCTION

Since the first diffraction of a crystal experiment was performed by Max von Laue in 1912, physical diffraction has been the most powerful tool for obtaining insights in the atomic structure of crystals.

The diffraction pattern of a fully periodic crystal consists of bright spots, called Bragg peaks, which appear at very precise locations: on the dual lattice to the lattice of periods of the crystal.

The diffraction of more general solids is usually a mixture of Bragg peaks and diffuse background (continuous spectrum), with random structures showing the trivial Bragg peak only.

For a long time it was believed that only periodic crystals can produce pure point spectra, viz. a diffraction pattern consisting exclusively of Bragg peaks. But in 1984, Shechtman et. all [24] reported the discovery of a solid with pure point diffraction and 5-fold symmetry, which is impossible in periodic crystals. Because of this discovery, the International Union of Crystallography redefined in 1991 the term of crystal to mean "any solid having an *essentially discrete* diffraction diagram" [12].

For an overview of the precise mathematical setup of physical diffraction we refer the reader to [3, Chapter 9] as well as to the articles [2, 11, 18, 19] (for background on the physical theory see also [10]). The diffraction pattern is described via the diffraction measure. This measure arises as the Fourier transform of the autocorrelation measure of the point set (or, more general, measure) which represents the model of the solid. By Lebesgue decomposition, the diffraction measure can be decomposed into a discrete part, corresponding to the Bragg spectrum, and a continuous part, corresponding to the continuous diffraction spectrum.

It is usually understood that the diffraction is essentially discrete if the set of Bragg peaks is relatively dense. In this context special subsets of Euclidean space have been prime examples. These sets were introduced

by Meyer in the 70s and later became known as Meyer sets. Indeed, the investigations of Meyer sets has been central to the topic of diffraction, see e.g. [20, 6, 4] and references therein. Meyer sets in Euclidean space do have a relatively dense set of Bragg peaks as had been suspected for a long time and was finally shown in [25]. Recent work of Kellendonk / Sadun [14] shows even a converse of some sort. More precisely, it gives that a dynamical system of Delone sets with finite local complexity is conjugate to a dynamical system of Meyer sets if it has a relatively dense set of continuous eigenvalues. In this sense, Meyer sets seem 'unavoidable' when one deals with sets with many Bragg peaks.

The main result of this note is Theorem 3.4 in Section 3. It provides another, somewhat surprising, instance of this unavoidability of Meyer sets. Namely, we show for ANY Delone set in Euclidean space that the set of Bragg peaks with high intensity is a Meyer set (provided it is relatively dense). Therefore, Meyer sets appear in a natural way also in the Bragg diffraction of any point sets with large sets of Bragg peaks of high intensity. If the underlying point set is Meyer itself the requirement of high intensity can be dropped as has already been shown in [26, 27]. This can be understood as saying that the class of Meyer sets is characterized by some form of selfduality under Fourier transform. In this spirit, we use our main result to give a new characterization for Meyer sets at the end of this note.

Our main result follows from a more general result dealing with positive and positive definite measures in Section 2. In fact, that section contains a study of positive and positive definite measures which may be of interest in its own right. The necessary background and notation is discussed in Section 1.

1. Background and notation

In this section we recall some basic concepts underlying our considerations. We are mainly interested in special subsets of Euclidean space (or, more generally, of a locally compact abelian group). However, our statements and proofs can be very conveniently phrased in the framework of measures. For this reason we introduce here both some background on measures and on sets.

For this entire section G denotes a locally compact abelian group (LCAG). We will denote by $C_{\mathsf{U}}(G)$ the space of uniformly continuous bounded function on G, which is a Banach space with respect to $\| \cdot \|_{\infty}$ norm. By $C_{\mathsf{c}}(G)$ we denote the subspace of $C_{\mathsf{U}}(G)$ of all compactly supported functions. We define the convolution $f * g$ of $f, g \in C_{\mathsf{c}}(G)$ by

$$f * g : G \longrightarrow \mathbb{C}, x \mapsto \int_G f(x-y)g(y)dy,$$

where dy denotes integration with respect to the Haar measure on G. Then, $f * g$ can easily be seen to belong to $C_{\mathsf{c}}(G)$ as well. Moreover, for any complex-valued function f on G we define the function $\widetilde{f}$ on G via $\widetilde{f}(x) = \overline{f(-x)}$.

The space $C_{\mathsf{c}}(G)$ is made into a locally convex space by the inductive limit topology, as induced by the canonical embeddings

$$C_K(G) \hookrightarrow C_{\mathsf{c}}(G), \quad K \subset G \text{ compact.}$$

Here, $C_K(G)$ is the space of complex valued continuous functions on G with support in K, which is equipped with the usual supremum norm. In line with the Riesz-Markov theorem, for us a measure μ on G will then be a linear functional on $C_{\mathsf{c}}(G)$, which is continuous with respect to the inductive topology on $C_{\mathsf{c}}(G)$, see [5, 22] for details.

The convolution of a measure μ with an $f \in C_{\mathsf{c}}(G)$ is defined as

$$\mu * f : G \longrightarrow \mathbb{C}, \ \mu * f(x) = \mu(f(x - \cdot)).$$

As is well known (see e.g. [22, Thm. 6.5.6] together with its proof), every measure μ gives rise to a unique measure $|\mu|$ called the *total variation* of μ, satisfying

$$|\mu|(f) \ = \ \sup \left\{ |\mu(g)| : g \in C_{\mathsf{c}}(G, \mathbb{R}) \text{ with } |g| \le f \right\}$$

for every non-negative $f \in C_{\mathsf{c}}(G)$. The total variation is a positive measure i.e. it maps non-negative functions to non-negative values and allows for the usual integration theory. Moreover, by [22, Thm. 6.5.6], there exists a measurable function $u : G \longrightarrow \mathbb{C}$ with $|u(t)| = 1$ for $|\mu|$-almost every $t \in G$ such that

$$\mu(f) \ = \ \int_G f \, u \, \mathrm{d}|\mu| \quad \text{for all } f \in C_{\mathsf{c}}(G).$$

We can use this to define for any measure μ on G and any bounded measurable function h on G the measure $h\mu$ by

$$(h\mu)(f) := \int_G f \, h \, u \, \mathrm{d}|\mu|$$

for $f \in C_{\mathsf{c}}(G)$. We can then also define the *discrete part* of a measure μ by

$$\mu_{\mathsf{pp}} = \sum_{p \in P} u(p)\delta_p,$$

where, for $q \in G$, we define the measure δ_q via $\delta_q(f) = f(q)$ and P is the set of those $q \in G$ such that $|\mu|(g) \geq 1$ whenever $g \in C_{\mathsf{c}}(G)$ is non-negative with $g(q) = 1$.

The measure μ on G is called *translation bounded* if for each compact set K we have

$$\sup_{x \in G} |\mu|\, (x + K) \, < \, \infty.$$

As mentioned already, besides measures and functions certain subsets of G with additional properties are the main object in our considerations. The corresponding pieces of notation are introduced next.

A subset of G is called *uniformly discrete* if there exists an open set V in G containing the neutral element of G such that $(x + V) \cap (y + V) = \emptyset$ whenever x and y are two different elements of the subset. A subset of G is called *relatively dense* if there exists a compact K such that any translate of K intersects the subset. A subset of G which is both uniformly discrete and relatively dense is called *Delone*. A subset of G is called *weakly uniformly discrete* if for any compact $K \subset G$ there is a C such that any translate of K meets at most C points of Γ. One can identify a weakly uniformly discrete set Λ in G with a measure by considering its *Dirac comb*

$$\delta_\Lambda := \sum_{x \in \Lambda} \delta_x.$$

In this way, all considerations below dealing with measures naturally descend to weakly uniformly discrete sets and in particular to Delone sets.

2. A study of positive and positive definite measures

We will be interested in measures and functions with additional positivity properties. More specifically, we will be interested in positive definite measures. We will present a study of certain features. As a consequence we will derive three main properties at end of this section.

Definition 2.1. Let G be an LCAG.
- The measure μ on G is *positive definite* if for all $f \in C_{\mathsf{c}}(G)$ we have $\mu(f * \widetilde{f}) \geq 0$
- The function $f : G \longrightarrow \mathbb{C}$ is *positive definite* if for all $N \in \mathbb{N}$ and $x_1, \ldots, x_N \in G$, the matrix $(f(x_k - x_l))_{k,l=1,\ldots,N}$ is positive definite.

It is well known (see e.g. [2]) that a measure μ is positive definite if and only if $\mu * (f * \widetilde{f})$ is positive definite for all $f \in C_{\mathsf{c}}(G)$. Also, we have the following well known result ([2]).

Proposition 2.2 (Krein's inequality for functions)**.** *Let G be an LCAG. Let f be a positive definite function on G. Then $f(0) \geq |f(x)|$ for all $x \in G$ and*

$$|f(x + t) - f(x)|^2 \leq 2f(0)[f(0) - \Re(f(t))]$$

for all $x, t \in G$ (where $\Re$ denotes the real part).

We start by showing that the restriction to the pure point part preserves positivity and positive definiteness.

Lemma 2.3. *Let G be an LCAG. Let μ be measure on G and μ_{pp} its discrete part.*

(a) *If μ is positive then μ_{pp} is positive.*
(b) *If μ is positive definite then μ_{pp} is positive definite.*
(c) *If μ is positive and positive definite then μ_{pp} is positive and positive definite.*

Proof: (a) is obvious. (b) follows from [17, Thm. 10.2]. (c) is an immediate consequence of (a) and (b). $\qquad\square$

We are now heading towards a Krein inequality for measures. To establish it we will need some preparation. We let G_{d} be the group G equipped with the discrete topology. Consider now a discrete measure μ on G. Then, μ can be identified with a measure on G_{d}.

Also, μ defines a function on G via $f(x) := \mu(\{x\})$. We call it the *support function* of μ. In the next Proposition 2.4 we show that the positive definiteness of μ as measure on G respectively G_{d} and of f as function on G respectively G_{d} are equivalent. This will allow us to translate properties of positive definite functions to discrete measures.

Proposition 2.4. *Let G be an LCAG. Let μ be a discrete measure on G and let $f : G \to \mathbb{C}$ be its support function*

$$f(x) := \mu(\{x\}).$$

Then the following assertions are equivalent:

(i) *μ is a positive definite measure on G.*
(ii) *μ is a positive definite measure on G_{d}.*
(iii) *f is a positive definite function on G.*
(iv) *f is a positive definite function on G_{d}.*

Proof: The equivalence $(i) \iff (ii)$ is [17, Thm. 10.1]. The equivalence $(iii) \iff (iv)$ follows immediately from the definition of positive definiteness (as the underlying topology is not relevant for the definition). To complete the proof we show that (ii) and (iv) are equivalent.

We first prove the $(iv) \implies (ii)$: As f is a positive definite function on G_{d} and the Haar measure $\theta_{G_{\mathsf{d}}}$ is a positive definite measure on G_{d}, it follows from [1, Cor. 4.3] that $f\theta_{G_{\mathsf{d}}}$ is a positive definite measure on G_{d}. As $\mu = f\theta_{G_{\mathsf{d}}}$, this proves (ii).

We next prove $(ii) \implies (iv)$: Since μ is a positive definite measure on G_{d}, it follows that $\mu * g * \widetilde{g}$ is a positive definite function for all $g \in C_{\mathsf{c}}(G_{\mathsf{d}})$. Let us observe that $g \in C_{\mathsf{c}}(G_{\mathsf{d}})$ if and only if g has a finite support. Therefore

$$g(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases} \in C_{\mathsf{c}}(G_{\mathsf{d}}).$$

Our claim now follows from the observation that with this choice of g we have $f = \mu * g * \widetilde{g}$. $\hspace{1cm}\square$

As a consequence of Proposition 2.4 we obtain the following version of Krein's inequality for positive definite discrete measures.

Corollary 2.5. (**Krein's inequality for measures**) *Let μ be a positive definite measure on G. Then all $x, t \in G$, we have*

$$\left|\mu(\{x+t\}) - \mu(\{x\})\right|^2 \leq 2\mu(\{0\})\left[\mu(\{0\}) - \Re(\mu(\{t\}))\right].$$

Proof. By Lemma 2.3 the measure μ_{pp} is positive definite. By the previous proposition its support function f is positive definite. Now, the corollary follows directly from Proposition 2.2 applied to f. $\hspace{1cm}\square$

We are now going to derive three main consequences of the previous considerations.

Our next result shows that if μ is a positive and positive definite measure on G, the set of points of measure close to $\mu(\{0\}) \neq 0$ has some sparseness property. This will be a main ingredient in our study of Meyer sets later on.

Lemma 2.6 (Sparseness Lemma). *Let G be an LCAG. Let μ be a positive and positive definite translation bounded measure and chose $a > (\sqrt{3} - 1)\mu(\{0\})$ and let*

$$I := \left\{x \in G \,|\, \mu(\{x\}) \geq a\right\}.$$

Then $I - I$ is weakly uniformly discrete.

Remark. Let us note that $\mu(\{0\})$ is greater than 0 whenever μ is not the zero measure (as can easily be inferred from Krein's inequality).

Proof: We prove first that there exists some $b > 0$ depending only on a and $\mu(\{0\})$) such that for all $x, y \in I$ we have $\mu(\{x - y\}) > b$. By Krein's Inequality we have

$$\left|\mu(\{x\}) - \mu(\{x-y\})\right|^2 \leq 2\mu(\{0\})\left[\mu(\{0\}) - \Re(\mu(\{y\}))\right].$$

Therefore, as μ is positive (and hence real), we have

$$\mu(\{x-y\}) \geq \mu(\{x\}) - \sqrt{2\mu(\{0\})\left[\mu(\{0\}) - \Re(\mu(\{y\}))\right]}$$
$$\geq a - \sqrt{2\mu(\{0\})\left[\mu(\{0\}) - a\right]}.$$

Let $b = a - \sqrt{2\mu(\{0\})\left[\mu(\{0\}) - a\right]}$. A short computation shows that $b > 0$ is equivalent to the condition on a in the statement of the proposition. Indeed,

$$b > 0 \iff a > \sqrt{2\,\mu(\{0\})\big[\mu(\{0\}) - a\big]}$$
$$\iff a^2 > 2\,\mu(\{0\})\big[\mu(\{0\}) - a\big]$$
$$\iff a^2 + 2\,\mu(\{0\})a + \big(\mu(\{0\})\big)^2 > 3\big(\mu(\{0\})\big)^2$$
$$\iff \big[a + \mu(\{0\})\big]^2 > 3\big(\mu(\{0\})\big)^2$$
$$\iff a > \mu(\{0\})(\sqrt{3} - 1).$$

Now, let

$$J := \big\{x \in G \,|\, \mu(\{x\}) \geq b\big\}.$$

We proved above that $I - I \subset J$. Thus, to complete the proof it suffices to show that J is weakly uniformly discrete.

Let $K \subset G$ be any compact set. Since μ is translation bounded, we have by definition

$$C := \sup_{t \in G} \mu(t + K) < \infty.$$

We show that for all $t \in G$ we have

$$\sharp\big((t + K) \cap J\big) \leq \frac{C}{b},$$

(which clearly proves that J is weakly uniformly discrete). Indeed, for all $t \in G$ we have

$$C \geq \mu(t + K) \geq \sum_{x \in \big((t+K) \cap J\big)} \mu(\{x\}) \geq b\sharp\big((t + K) \cap J\big).$$

This shows that J is weakly uniformly discrete and the proof is finished. $\square$

Our next two results show that a relatively dense set can only give rise to a positive definite Dirac comb if it is a lattice. This ties in well with various recent strings of research (see remark below).

Lemma 2.7 (Rigidity Lemma). *Let G be an LCAG. Let $\Lambda \subset G$ be weakly uniformly discrete. Then δ_Λ is positive definite if and only if Λ is discrete subgroup of G.*

Proof: The implication '$\Longleftarrow$' is obvious. It remains to prove the implication '$\Longrightarrow$'. As δ_Λ is positive definite, it follows from Proposition 2.4 that the function

$$f(x) = \begin{cases} 1, & \text{if } x \in \Lambda, \\ 0, & \text{otherwise.} \end{cases},$$

is positive definite. As $f \not\equiv 0$ (as Λ is relatively dense), it follows that $f(0) \neq 0$, and hence $f(0) = 1$. Let now $x, y \in \Lambda$ be arbitrary. Then, by Krein's inequality we have:

$$\big|f(x) - f(x - y)\big|^2 \leq 2f(0)\big[f(0) - \Re(f(y))\big] = 2[1 - 1] = 0.$$

Therefore $f(x - y) = f(x)$. As $f(x) = 1$ it follows that $f(x - y) = 1$ and hence $x - y \in \Lambda$. This shows that

$$\Lambda - \Lambda \subset \Lambda,$$

and thus Λ is a subgroup of G. As Λ is weakly uniformly discrete it follows that it must even be uniformly discrete and it follows that Λ is a discrete subgroup in G. $\qquad\square$

Corollary 2.8. *Let $\Lambda \subset G$ be a Delone set. Then δ_Λ is positive definite if and only if Λ is a lattice.*

Proof. By the previous lemma, Λ is a discrete subgroup. By assumption it is furthermore relatively dense and uniformly discrete. Thus, it is a lattice. $\quad\square$

Remark. The result can be seen in the context of a famous theorem of Cordoba [8] and a well-known question of Lagarias [15]. The theorem of Cordoba says that if Λ is a Delone set, and δ_Λ is Fourier transformable with discrete Fourier transform, then Λ is crystallographic (i.e. a finite union of translates of the same lattice). The question of Lagarias asks whether every Delone set Λ with strongly almost periodic Dirac comb δ_Λ is actually crystallographic. Note that if δ_Λ is Fourier transformable with discrete Fourier transform, then δ_Λ is a strong almost periodic measure [17]. Recently a positive answer to Lagarias question was given under the additional hypothesis of finite local complexity independently in [9] and [13]. Moreover, in [13] it is shown that the answer is in general negative without the assumption of finite local complexity. In the context of Meyer sets corresponding results were already obtained in [26].

Another simple consequence of Proposition 2.4 is the fact that given a discrete positive definite measure, its restriction to a closed subgroup of G is also positive definite. In the remainder of this section we investigate when the restriction to a subgroup preserves the positive definiteness. We start by defining the restriction of a measure to a subgroup.

Definition 2.9. Let G be an LCAG. Let μ be a measure on G and let H be a closed subgroup of G. We define the restriction of μ to H by

$$\mu|_H(B) := \mu(B \cap H).$$

for B a Borel set in G. Then $\mu|_H$ is a measure on G with $\mathrm{supp}(\mu|_H) \subset H$, and can therefore be seen as a measure on H.

Note that since H is closed in G, the characteristic function 1_H is measurable and locally integrable. It is easy to see that $\mu|_H = 1_H \mu$.

Lemma 2.10 (Restriction lemma - first version). *Let μ be a discrete positive definite measure on G, and let H be a closed subgroup of G. Then $\mu|_H$ is a positive definite measure on H.*

Proof: We denote by G_{d} and H_{d}, respectively, the groups G and H when equipped with discrete topology. Let $f : G \to \mathbb{C}$ be the support function of μ given by $f(x) = \mu(\{x\})$. Then, by Proposition 2.4, f is a positive definite function on G_{d}. This directly gives that the restriction $g : H \to \mathbb{C}, g(x) = f(x)$ is a positive definite function on H_{d}, and hence again by Proposition 2.4 the measure $\sum_{x \in H_{\mathsf{d}}} g(x)\delta_x$ is positive definite measure on H. But this is exactly the desired statement. $\qquad\square$

Combining Proposition 2.4 with Corollary 2.10 we get the following generalization of [3, Lemma 8.4]:

Corollary 2.11. *Let L be a lattice in G, let $\eta : L \to \mathbb{C}$ be a function and let $\mu = \eta\delta_L$. Then μ is a positive definite measure on G if and only if η is a positive definite function on L.*

If H is an open subgroup of G, it is automatically closed. In this case, it follows immediately from $C_{\mathsf{c}}(H) \subset C_{\mathsf{c}}(G)$ that the restriction of any positive definite measure on G to H is a positive definite measure on H.

Lemma 2.12 (Restriction lemma - second version). *Let G be an LCAG. Let μ be a positive definite measure on G, and let H be a open subgroup of G. Then $\mu|_H$ is a positive definite measure on H.*

Proof: As H is open in G, we have $C_{\mathsf{c}}(H) \subset C_{\mathsf{c}}(G)$. Therefore, for all $f \in C_{\mathsf{c}}(H)$ we have

$$\mu|_H(f * \widetilde{f}) = \mu(f * \widetilde{f}) \geq 0,$$

with the first equality follows from the fact that the support of $f * \widetilde{f}$ is contained in H and the second equality follows as f belongs to $\in C_{\mathsf{c}}(G)$ as well. $\qquad\square$

3. On relatively dense sets of a-visible Bragg peaks

In this section we restrict our attention to positive and positive definite measures in $\mathbb{R}^d$. We will combine our previous considerations with certain ingredients from mathematical diffraction theory to obtain the Meyer property for certain subsets of the set of Bragg peaks and to provide a new characterization of the Meyer property.

We will be interested in Meyer sets. There are various characterizations of Meyer sets in Euclidean space (see e.g. [16, 20, 21]). Here, we will use that a subset Γ of $\mathbb{R}^d$ is *Meyer* if and only if Γ is relatively dense and $\Gamma - \Gamma$ is weakly uniformly discrete. A more common definition requires that Γ is relatively dense and $\Gamma - \Gamma$ is uniformly discrete. However, based on [16] these two definitions are shown to be equivalent in the appendix of [4].

Next we will review the theory of mathematical diffraction. For overviews of this theory we refer the reader to [2, 18, 19]. During the entire section

$\{A_n\}_n$ will be a *van Hove sequence* in $\mathbb{R}^d$ i.e. the A_n will be relatively compact subsets of $\mathbb{R}^d$ with

$$|\partial^R A_n|/|A_n| \to 0, n \to \infty$$

for all $R > 0$. Here, $|\cdot|$ denotes Lebesgue measure and, for $B \subset \mathbb{R}^d$, the set $\partial^R B$ consists of all $x \in \mathbb{R}^d$ whose distance from both B and $\mathbb{R}^d \setminus B$ does not exceed R. Obviously, any sequence of balls (cubes) with radius (sidelength) tending to ∞ is a van Hove sequence. For a translation bounded measure ω on $\mathbb{R}^d$ we define

$$\gamma_n := \frac{\omega|_{A_n} * \widetilde{\omega|_{A_n}}}{|A_n|}.$$

Here, for a measure ν we denote by $\nu|_A$ the restriction of ν to A and by $\widetilde{\nu}$ the measure with $\widetilde{\nu}(f) = \overline{\nu(\widetilde{f})}$.

Definition 3.1. Let ω be a translation bounded measure. Any cluster point γ of the sequence $(\gamma_n)_n$ in the vague topology is called *an autocorrelation* of ω.

Remark. Let ω be a translation bounded measure in $\mathbb{R}^d$. Let U be an open relatively compact subset of $\mathbb{R}^d$. Then, $C := \sup |\omega|(x + U) < \infty$. As shown in [5] the space $\mathcal{M}_{U,C}$ of translation bounded measures μ on $\mathbb{R}^d$ with $\sup |\mu|(x + U) \leq C$ is compact in the vague topology and all γ_n belong to this space. It follows that the sequence γ_n always has cluster points.

As is well-known (and not hard to see) any autocorrelation γ of a translation bounded ω is positive definite. For this reason its Fourier transform $\widehat{\gamma}$ exists and is a positive measure on the dual group $\widehat{\mathbb{R}^d}$ of $\mathbb{R}^d$, see [17, 5] for further discussion. We call this Fourier transform *a diffraction measure* for ω. We define the autocorrelation of a Delone set $\Lambda \subset \mathbb{R}^d$ to be the autocorrelation of its Dirac comb $\delta_\Lambda = \sum_{x \in \Lambda} \delta_x$.

Let us recall next the definition of a-visible Bragg peaks, see [26] as well.

Definition 3.2. Let μ be a translation bounded measure on $\mathbb{R}^d$, and let γ be any autocorrelation of μ. For each $a > 0$ we call

$$I(a) := \{\chi \in \widehat{\mathbb{R}^d} \,|\, \widehat{\gamma}(\{\chi\}) \geq a\}$$

the set of *a-visible Bragg peaks of μ*.

After this review of diffraction theory we now note the following consequence of the Sparseness Lemma 2.6.

Lemma 3.3. *Let μ be a positive and positive definite translation bounded measure on $\mathbb{R}^d$. If the set*

$$I := \left\{ x \in \mathbb{R}^d \,|\, \mu(\{x\}) \geq a \right\}.$$

is relatively dense for some $a > (\sqrt{3} - 1)\mu(\{0\})$, then I is a Meyer set.

Proof: By Lemma 2.6 the set $I - I$ is weakly uniformly discrete. As I is also relatively dense, the statement follows. $\qquad\square$

Remark. A natural question is if the lower bound $(\sqrt{3} - 1)\mu(\{0\})$ can be improved in Lemma 3.3. We provide an example which shows that it cannot be decreased under $\frac{1}{2}\mu(\{0\})$: Let $\mu = \delta_{\mathbb{Z}} + \delta_{\pi\mathbb{Z}}$. Then for all $a > 1 = \frac{1}{2}\mu(\{0\})$ the set

$$\{x \in \mathbb{R} \,|\, \mu(\{x\}) \geq a\} = \{0\}\,.$$

is not relatively dense. However, in the case $a = 1$ the set

$$I := \{x \in \mathbb{R} \,|\, \mu(\{x\}) \geq a\} = \mathbb{Z} \cup \pi\mathbb{Z}\,,$$

is relatively dense and

$$I - I = \mathbb{Z} \oplus \pi\mathbb{Z}\,,$$

which is dense in $\mathbb{R}$.

We are now proceeding to prove our main result.

Theorem 3.4. *(a) Let μ be a positive translation bounded measure on $\mathbb{R}^d$ and let γ be an autocorrelation of μ. If the set $I(a)$ of a visible Bragg peaks of μ is relatively dense for some $a > (\sqrt{3} - 1)\widehat{\gamma}(\{0\})$, then $I(a)$ is a Meyer set.*

(b) Let Λ be a Delone set in $\mathbb{R}^d$ and let γ be an autocorrelation of Λ. If the set $I(a)$ of a visible Bragg peaks of μ is relatively dense for some $a > (\sqrt{3} - 1)\widehat{\gamma}(\{0\})$, then for all $(\sqrt{3} - 1)\widehat{\gamma}(\{0\}) < b \leq a$ the set $I(b)$ is a Meyer set.

Proof: (a) If μ is a positive translation bounded measure on $\mathbb{R}^d$, any autocorrelation γ is positive and positive definite. Therefore, so is $\widehat{\gamma}$ [1], [7]. Therefore, applying the result of Lemma 3.3 to $\widehat{\gamma}$ we obtain the first statement.

(b) This follows from (a) as $I(a) \subset I(b)$ if $(\sqrt{3} - 1)\widehat{\gamma}(\{0\}) < b \leq a$. $\qquad\square$.

Remark. Let us put the previous result in perspective.

- In Theorem 3.4, if some $I(a)$ is relatively dense, then $I(b)$ is relatively dense for all $b < a$. The same is not necessarily true for $b > a$ as can be easily seen by considering a variant of the example given in the remark following Lemma 3.3 above. In fact, the mentioned example has the desired property (but is not a Delone set). To obtain a similar feature with a Delone set, we can consider

$$\Lambda = \mathbb{Z} \times \mathbb{Z} \cup \left(\left(\frac{1}{2}, 0\right) + \mathbb{Z} \times (\pi\mathbb{Z})\right).$$

- This result can be compared with a corresponding result when the underlying set is Meyer itself. Then, for each $0 < a < \widehat{\gamma}(\{0\})$ the set

$$I(a) = \{\chi \in \widehat{\mathbb{R}^d} \,|\, \widehat{\gamma}(\{\chi\}) \geq a\}\,,$$

of a-visible Bragg peaks is Meyer [26, 27].

We finish this section by using the preceding results to provide a new characterization of Meyer sets in terms of positive definite measures.

Theorem 3.5. *Let $\Lambda \subset \mathbb{R}^d$ be relatively dense. Then, the following assertions are equivalent.*

(i) *Λ is a Meyer set.*

(ii) *For each $0 < \varepsilon < 1$ there exists a positive and positive definite measure μ such that, for all $x \in \Lambda$ we have*

$$\mu(\{x\}) > \varepsilon \mu(\{0\}).$$

(iii) *There exists a positive and positive definite measure μ and some $\sqrt{3} - 1 < \epsilon < 1$ such that, for all $x \in \Lambda$ we have*

$$\mu(\{x\}) > \varepsilon \mu(\{0\}).$$

(iv) *For each $0 < \varepsilon < 1$ there exists a Meyer set $\Gamma \subset \widehat{\mathbb{R}^d}$, with autocorrelation γ, such that*

$$\Lambda \subset I(\varepsilon \widehat{\gamma}(\{0\})),$$

where $I(\varepsilon \widehat{\gamma}(\{0\}))$ is the set of $\varepsilon \widehat{\gamma}(\{0\})$-visible peaks of Γ.

(v) *There exists some $0 < \varepsilon < 1$ and a Meyer set $\Gamma \subset \widehat{\mathbb{R}^d}$, with autocorrelation γ, such that*

$$\Lambda \subset I(\varepsilon \widehat{\gamma}(\{0\})).$$

(vi) *For each $0 < \varepsilon < 1$ there exists a Delone set $\Gamma \subset \widehat{\mathbb{R}^d}$, with autocorrelation γ, such that*

$$\Lambda \subset I(\varepsilon \widehat{\gamma}(\{0\})).$$

(vii) *There exists some $\sqrt{3} - 1 < \varepsilon < 1$ and a Delone set $\Gamma \subset \widehat{\mathbb{R}^d}$, with autocorrelation γ, such that*

$$\Lambda \subset I(\varepsilon \widehat{\gamma}(\{0\})).$$

Proof: For any subset Σ of $\mathbb{R}^d$ and any $\varepsilon > 0$ we define

$$\Sigma^\varepsilon := \{\chi \in \widehat{\mathbb{R}^d} : |\chi(x) - 1| < \varepsilon \text{ for all } x \in \Sigma\}.$$

Similarly, we define for any subset Ξ of $\widehat{\mathbb{R}^d}$ and any $\varepsilon > 0$

$$\Xi^\varepsilon := \{x \in \mathbb{R}^d : |\chi(x) - 1| < \varepsilon \text{ for all } \chi \in \Sigma\}.$$

We will use below that Meyer sets can be characterized via these sets.

The implications $(ii) \implies (iii)$, $(iv) \implies (v)$ and $(iv) \implies (vi) \implies (vii)$ are obvious, while $(iii) \implies (i)$ follows from Lemma 3.3. The implication $(vii) \implies (i)$ follows from Theorem 3.4. $(iv) \implies (ii)$ follows from the fact that $\mu = \widehat{\gamma}$ is positive and positive definite [17, 7]. $(v) \implies (i)$ follows from [26, Thm 5.3 (iii)]. To complete the proof we prove $(i) \implies (iv)$.

Let $\varepsilon \in (0,1)$ and let $\varepsilon' = 1 - \varepsilon$ and $\Lambda' = \left(\Lambda^{\frac{\varepsilon'}{2}} \right)^{\frac{\varepsilon'}{2}}$. Then $\Lambda \subset \Lambda'$ and $\left((\Lambda')^{\frac{\varepsilon'}{2}} \right)^{\frac{\varepsilon'}{2}} = \Lambda'$ [20]. Let $\Gamma = (\Lambda')^{\frac{\varepsilon'}{2}}$. We prove that Γ has the desired property. Let γ be an autocorrelation of Γ. Then $\mathrm{supp}(\gamma) \subset \Gamma - \Gamma =: \Delta$. Therefore, by [26, Thm. 3.1], for all $y \in \Delta^{\varepsilon'}$ and all $x \in \mathbb{R}^d$ we have

$$\left| \widehat{\gamma}(\{x + y\}) - \widehat{\gamma}(\{x\}) \right| \leq \varepsilon' \widehat{\gamma}(\{0\}).$$

Now, by [20, proof of Cor. 6.8] we have

$$\Gamma^{\frac{\varepsilon'}{2}} \subset (\Gamma - \Gamma)^{\varepsilon'} = \Delta^{\varepsilon'}.$$

This implies

$$\Lambda \subset \Lambda' \subset \Gamma^{\frac{\varepsilon'}{2}} \subset \Delta^{\varepsilon'}.$$

Therefore, for all $y \in \Lambda \subset \Delta^{\varepsilon'}$, we have

$$\left| \widehat{\gamma}(\{y\}) - \widehat{\gamma}(\{0\}) \right| \leq \varepsilon' \widehat{\gamma}(\{0\}).$$

This gives

$$\widehat{\gamma}(\{y\}) \geq \left(1 - \varepsilon' \right) \widehat{\gamma}(\{0\}) = \varepsilon \widehat{\gamma}(\{0\}),$$

which finishes the proof. $\qquad\square$

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